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Exact solutions for nonconservative Broadwell–Boltzmann models

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Abstract. Extended discrete kinetic theory including sources, sinks, creation and annihilation of test particles, inelastic scattering, etc added to the elastic collisions, has recently been introduced, in which, for instance polynomials of the mass are added to the mass conservation law which becomes nonconservative. Here, to the classical two- and three-dimensional Broadwell elastic collision models, we associate nonconservative models adding, to the conservation law, a polynomial of the second degree. We present a method, for the determination of similarity, periodic and $(1+1)$-dimensional exact solutions with one space variable, that we apply to the Broadwell models with Piechor–Platkowski quadratic nonconservative terms.

1. Introduction

Recently [1, 2] an extended discrete kinetic theory (that we call nonconservative), introduced by Boffi and Spiga, has been extensively studied. It is written in the literature *the study of a gas with only elastic scattering between the molecules seems to be too idealized for application to the real world*. So they add a background medium, external sources and sinks, effects of absorption, and generation due to inelastic scattering, etc. A great difference, with conservative discrete kinetic theory, is that the conservation laws are modified by including polynomial functions of the densities. However, while for conservative discrete models [3] the determination of exact solutions is well understood [4], for nonconservative models exact solutions are missing. These nonconservative models do not have the important difficulties of the conservative ones [3]. For instance, the two-velocity models satisfying two nonlinear equations where only one linear combination can exist and necessarily the linear momentum conservation law is violated. For models with at most three nonlinear equations (such as the present Broadwell models) only two linear conservation laws can be deduced and the mass and energy conservation laws cannot be distinguished. For models with more nonlinear equations it can occur that not only linear physical conservation laws exist but others with the appearance of 'spurious' conservation laws. Conversely for conservative models [3] the existence of the conservation laws can be useful. For instance for shock waves, the application of the Lax criterion, the determination of the characteristic velocities, the sound waves, the supersonic and subsonic inequalities, etc.

For the exact solutions of conservative models the existence of linear conservation laws is important. First for the similarity solutions (variable $x-\xi t$, space x, time t), introducing the properties coming from the linear differential conservation law, we are reduced to a system of coupled Riccati solutions. When in addition to the linear conservation law there exists only one nonlinear equation. Like for the one-dimensional two velocity models or the two- and three-dimensional Broadwell models with solutions depending on one spatial coordinate, we only have one scalar Riccati equation which is integrable [5]. When two coupled Riccati equations remain, it implies that two classes of solutions have been found: either compatibility between two scalar Riccati equations which give monotonic solutions or solutions more complicated which are nonmonotonic [6]. When three nonlinear equations remain, it implies that only solutions compatible with three scalar Riccati equations have been recently found [4]. For the present nonconservative Broadwell models, without linear conservative relations, we have three quasilinear partial differential equations (PDEs) with quadratic nonlinearities. We will present solutions arising from the compatibility of three scalar Riccati equations and give a method which, in principle, could be applied to other nonconservative model with four, five, etc quasilinear PDEs. Secondly let us consider periodic, $(1+1)$ and $(2+1)$-dimensional solutions. For conservative models they are linear combinations of similarity solutions (with pairs complex conjugate for periodic parts), which, from the superposition principle, are automatically satisfied for the linear differential conservation laws. For the remaining nonlinear equations we obtain compatibility conditions between the loss and gain collision terms.

Now for nonconservative models the modified conservation equations do not have gain and loss quadratic terms and we cannot obtain solutions which are sums of similarity waves. Here most of the solutions will be obtained from the resolution of one linear PDE.

We wish to explain, for the determination of possible similarity and $(1+1)$ solutions to the nonconservative models, a simple method so that the reader can try to extend these solutions for other quasilinear PDE models with quadratic nonlinearities. However, as we shall explicitly see here for the Broadwell models, many constraint relations have to be satisfied. For instance, it is not easy to predict in advance whether positivity properties will be satisfied. Our main idea is as follows: Consider quadratic nonlinearities built up with densities $N_i(x,t)$ associated to linear differential terms. From the vanishing of these quadratic nonlinearities, we can hope to obtain for each N_i two constant values n_{0i} and $n_{0i}+n_i$ leading to two different identities. Let us consider the identities associated to the set $n_{0i}+n_i$. We can eliminate the terms coming from the identities associated to n_{0i}. Identities remain with only n_i terms linear and quadratic and they are opposite. Assuming now $N_i(x,t)=n_{0i}+n_iN(x,t)$ then the quadratic nonlinearities will be written $N(N-1)$ multiplied by constants which are quadratic in the n_i set. Finally, the linear differential terms will depend only on N derivatives and we will have to verify whether these $N(N-1)$ quadratic terms and linear differential N terms are compatible.

Here we wish to present our first two assumptions in a general framework. In sections 2 and 3 they are directly explained for the Broadwell models.

For $i=1,2\ldots p$ independent $N_i(x,t)$ we consider a quasilinear system of $j=1,2\ldots p$ coupled equations with constant real coefficients $a_i^{(j)}, b_i^{(j)}, c_i^{(j)}, c_{ik}^{(j)}, d_i^{(j)}$ and two variables x,t:

$$\begin{aligned} &L_j \equiv R_j \qquad L_j = \sum_{i=1}^{p}(a_i^{(j)}\partial_x + b_i^{(j)}\partial_t)N_i \\ &R_j = \sum_{i,k=1}^{p}[c_i^{(j)}N_i^2 + c_{ik}^{(j)}N_iN_k + d_i^{(j)}N_i + e_i^{(j)}]. \end{aligned} \tag{1.1}$$

Our first assumption is that from the set $R_j=0$ with $N_i=n_{as,i}$, $i=1,\ldots p$ we can obtain for any N_i at least two real roots $n_{as,i}=n_{0i}$, $n_{0i}+n_i$. For similarity solutions these $n_{as,i}$ will be asymptotic states and for Broadwell models they must be positive. For damped oscillating waves and $(1+1)$ solutions, at least one of them will be an asymptotic state.

$$\sum_{i,k=1}^{p} [c_i^{(j)} n_{as,i}^2 + c_{ik}^{(j)} n_{as,i} n_{as,k} + d_i^{(j)} n_{as,i} + e_i^{(j)}] = 0. \tag{1.2a}$$

We write (1.2a) for the set $n_{as,i} = n_{0i} + n_1$, eliminate the terms coming from $n_{as,i} = n_{0i}$, and deduce another relation which contains opposite terms quadratic and linear in n_i, n_k:

$$B_j := \sum c_i^{(j)} n_i^2 + c_{ik}^{(j)} n_i n_k = -\left[\sum c_i^{(j)} 2 n_{0i} n_i + c_{ik}^{(j)} (n_{0i} n_k + n_{0k} n_i) + d_i^{(j)} n_i \right]. \tag{1.2b}$$

Our second assumption is, for the $N_i(x,t)$, the existence of a common $N(x,t) = 1/D(x,t)$:

$$N_i(x,t) = n_{0i} + n_i N(x,t) \qquad N(x,t) D(x,t) = 1 \qquad n_{as,i} = n_{0i}, n_{0i} + n_i \tag{1.3}$$

giving the possibility to reduce the quasilinear (1.1) system to a linear one.

As above we substitute $N_i(x,t)$ given by (1.3) into (1.1) $R_j(x,t)$ and eliminate the constant terms coming from the set $n_{as,i} = n_{0i}$. Quadratic terms remain in $n_i N, n_k N$ which factorize N^2 and terms linear in $n_i N$ which factorize N and, due to (1.2b), they have opposite constants B_j. The R_j factorize $N(N-1)$ with linear PDE in (1.1):

$$\begin{gathered} R_j = B_j N(N-1) = B_j (D-1)/D^2 \qquad L_j = -\sum (n_i/D^2)(a_i^{(j)} \partial_x + b_i^{(j)} \partial_t)] D \\ \sum_{i=1}^{p} [n_i (a_i^{(j)} \partial_x + b_i^{(j)} \partial_t)] D(x,t) = B_j (D(x,t) - 1). \end{gathered} \tag{1.4}$$

The open nontrivial problem is whether or not compatibility conditions exist which satisfy the p system (1.4) with only one function $N(x,t)$ or $D(x,t)$.

In section 2 for similarity solutions functions of $z = x - \xi t$, we write explicitly the corresponding (1.4) compatibility conditions and our assumption is that they can be satisfied. Later we apply to the Broadwell models for which we are able to justify these assumptions and satisfy the compatibility and positivity conditions. In section 3, for periodic and $(1+1)$-dimensional solutions, we assume that only one N_k is x, t-dependent while the others are constants: $N_j = n_{0j}, j \neq k$. We write the compatibility conditions that we justify for the Broadwell models with positivity constraints. Two classes of solutions occur depending on whether one of the two $n_{as,k}$ or both are positive.

The two- and three-dimensional Broadwell models, with solutions depending on one spatial coordinate x, have three independent densities $N_i(x,t), i = 1, 2, 3$. The densities N_1, N_2 are associated to velocities ± 1 along the x-axis while N_3 (multiplicity $2(d-1)$) has zero for velocity projection along the x-axis. We define the total mass ρ, the elastic collision term Q with cross section σ and write the three nonconservative equations for the $N_j, j = 1, 2, 3$:

$$\begin{gathered} Q = N_3^2 - N_1 N_2 \qquad \rho = N_1 + N_2 + 2(d-1) N_3 \\ e_1 = -e_2 = 1 \qquad e_3 = 0 \qquad \sigma_1 = \sigma_2 = \sigma = -(d-1)\sigma_3 \\ L_j = (\partial_t + e_j \partial_x) N_j = R_j = \sigma_j Q + \kappa_j (\alpha \rho^2 + \eta \rho + \zeta) - N_j (\beta \rho + \epsilon) + S_j. \end{gathered} \tag{1.5}$$

Q is the only present term in the conservative models. The others are the nonconservative terms: $\beta \geqslant 0, \epsilon \geqslant 0$ ($\alpha \geqslant 0, \eta \geqslant 0, \zeta \geqslant 0$) for the annihilation (creation) of test particles as a result of inelastic collisions which can be quadratic or linear in the densities. The $\kappa_j \in (0,1)$ with $\kappa_1 + \kappa_2 + 2(d-1)\kappa_3 = 1$ represent the fractions of secondary particles generated with velocities $+1, -1, 0$ along the x-axis. As usual $S_i > 0$ (< 0) are constants associated to the external sources (sinks) which are independent of the densities.

There exists a second equivalent representation of (1.5), written in terms of ρ (mass), $j = N_1 - N_2$ (momentum along the x-axis) and N_3:

$$\begin{aligned} &L_\rho = \partial_t \rho + \partial_x j = R_\rho = (\alpha - \beta)\rho^2 + (\eta - \epsilon)\rho + \delta \\ &\delta := S_1 + S_2 + 2(d-1)S_3 + \zeta \\ &L_j = \partial_t j + \partial_x(\rho - 2(d-1)N_3) = R_j = (\kappa_1 - \kappa_2)(\alpha\rho^2 + \eta\rho + \zeta) \\ &\qquad\qquad - j(\beta\rho + \epsilon) + S_1 - S_2 \\ &Q = N_3^2 d(2-d) + (j^2 - \rho^2)/4 + (d-1)\rho N_3 \\ &L_3 = \partial_t N_3 = R_3 = -\sigma Q/(d-1) + \kappa_3(\alpha\rho^2 + \eta\rho + \zeta) - N_3(\beta\rho + \epsilon) + S_3. \end{aligned} \tag{1.6}$$

There exist nonconservative models [2] with quadratic ($\alpha \neq 0, \beta \neq 0$) ρ terms and another with only linear terms. We choose the Piechor–Platkowski quadratic models such that, for the vanishing r.h.s. in (1.5)–(1.6), two asymptotic states can exist for $N_i = n_{as,i}$ in (1.5) or $\rho_{as}, j_{as}, n_{as,3}$ in (1.6). Let us note the following scaling invariance. Let us define $\tilde{t} = t\epsilon, \tilde{x} = x\epsilon$ and consider the ratios of the creation, annihilation, sources and sinks, elastic cross section parameters $\alpha, \beta, \eta, \delta, \zeta, S_i, \sigma$ by ϵ while the fractions terms κ_i are fixed. Then (1.5) and (1.6) become invariant with $N_j, \rho, j \to N_j(\tilde{x}, \tilde{t})$, etc and we can always choose $\epsilon = 1$.

Like for conservative models, the r.h.s. of the mass (energy proportional to the mass for Broadwell models) and momentum (1.6) equations are independent of the cross section σ. For travelling waves, functions of $z = x - \xi t$, the two macroscopic asymptotic states ρ_{as}, j_{as} depend only on $\alpha, \beta, \eta, \epsilon, \delta, \kappa_j, S_1 - S_2$. Conversely from ρ_{as}, j_{as} it is interesting to study the associated microscopic quantities with the possible sources, sinks terms S_j, $j = 1, 2, 3$. The damped oscillating waves must have only one positive asymptotic state while for (1+1) solutions we must distinguish the models with only one or two positive asymptotic states.

2. Similarity solutions

2.1. General (1.1) system

We consider the similarity solutions functions of $z = x - \xi t$ with assumptions (1.2a) and (1.2b): existence of $n_{as,k} = n_{0k}, n_{0k} + n_k, k = 1, \ldots p$ coming from $(R_j(n_{as,1}, \ldots n_{as,p}) = 0$ and (1.3), common $N(z) = 1/D(z))$ leading to the compatibility conditions (1.4) that we rewrite for the scalar p Riccati equations (N) or linear ordinary differential equations (ODEs) (D):

$$\begin{aligned} &N_j(x,t) = n_{0i} + n_i/D(x,t) \qquad N(x,t)D(x,t) = 1 \qquad n_{as,j} = n_{0j}, n_{0j} + n_j \\ &A_j \mathrm{d}D/\mathrm{d}z = 1 - D \qquad A_j = \left[\sum(\xi b_i^{(j)} - a_i^{(j)})n_i\right] \Big/ \left[\sum c_i^{(j)} n_i^2 + c_{ik}^{(j)} n_i n_k\right]. \end{aligned} \tag{2.1}$$

From the second assumption D is j-independent and we will have to verify that a compatibility condition between these linear ODEs exists. If so, we integrate easily:

$$\begin{aligned} &A_1 = A_2 = \cdots = A_p = -1/\gamma \\ &\mathrm{d}N/\mathrm{d}(\gamma z) = N(N-1) \qquad D(z) = 1 + \bar{d}\mathrm{e}^{\gamma z} \qquad \bar{d} > 0 \end{aligned} \tag{2.2}$$

where $\bar{d}$ is an arbitrary constant which does not enter into the parameter relations. We obtain

$$\gamma z \to \pm\infty : D(z) \to +\infty, 1 \qquad N(z) \to 0, 1 \qquad N_i(z) \to n_{0i}, n_{0i} + n_i. \tag{2.3}$$

Let us assume that we are able to satisfy all the constraints coming from our three assumptions. For a study of the stability around the two $N(z)$ asymptotic states $0, 1$ we substitute $N(z) + \delta_N(z)$ into (2.3), linearize in $\delta_N(z)$ and obtain

$$\frac{\mathrm{d}\delta_N}{\mathrm{d}(\gamma z)} = (2N - 1)\delta_N$$
$$\gamma z \to \pm\infty :\to \mathrm{d}\delta_N/\mathrm{d}(\gamma z) = \mp\delta_N \qquad \text{and} \qquad \delta_N \to 0. \tag{2.4}$$

For a system of $p \geqslant 2$ equations, forgetting the model parameters, we have $2(p+1)$ parameters n_{0i}, n_i, γ, ξ coming from the solutions and $3p$ constraints (1.2*a*)–(2.2) which means that some model parameters must enter into the constraint relations. We will determine such solutions for the Broadwell nonconservative models with $p = 3$ and we expect to finally obtain a constraint between the model parameters. In view of the nonconservative model parameters of the (1.5)–(1.6) type $\alpha, \beta, \epsilon, \eta, \zeta, S_i, \kappa_i$ (in fact one is a scaling parameter) we have $2(p+2)$ additional parameters.

2.2. Similarity solutions for the nonconservative Broadwell models

Our first assumption is that for travelling waves we are able to find two positive asymptotic states for each density $N_k, k = 1, 2, 3$. We choose representation (1.6) and write the linear differential terms $L_\rho = R_\rho, L_j = R_j, L_3 = R_3$ for similarity solutions:

$$L_\rho = \frac{\mathrm{d}}{\mathrm{d}z}(-\xi\rho(z) + j(z)) \qquad L_j = \frac{\mathrm{d}}{\mathrm{d}z}(-\xi j(z) + \rho(z) - 2(d-1)N_3(z))$$
$$L_3 = -\xi\frac{\mathrm{d}}{\mathrm{d}z}N_3(z). \tag{2.5}$$

L_ρ, L_j, L_3 are zero for constant asymptotic states with $|z| \to \infty$, and we write with $R_\rho = R_j = R_3 = 0$ the $\rho_{as}, j_{as}, n_{as,3}$ asymptotic relations:

$$\rho_{as} = \rho_0, \rho_0 + \rho_1 : (\alpha - \beta)\rho_0^2 + (\eta - \epsilon)\rho_0 + \delta \qquad (\rho_1 + 2\rho_0)(\alpha - \beta) = \epsilon - \eta \tag{2.6a}$$

$$j_{as} = j_0, j_0 + j_1 : j_0(\beta\rho_0 + \epsilon) = (\kappa_1 - \kappa_2)(\alpha\rho_0^2 + \eta\rho_0 + \zeta) + S_1 - S_2$$
$$j_1(\beta(\rho_0 + \rho_1) + \epsilon) = \rho_1(\kappa_1 - \kappa_2)(\alpha(\rho_1 + 2\rho_0) + \eta) - \beta j_0\rho_1 \tag{2.6b}$$

$$n_{as,3} = n_{03}, n_3 + n_{03} : \sigma[d(2-d)n_{as,3}^2 + (j_{as}^2 - \rho_{as}^2)/4]/(d-1)$$
$$= -n_{as,3}[\rho_{as}(\sigma + \beta) + \epsilon] + \kappa_3[\alpha\rho_{as}^2 + \eta\rho_{as} + \zeta] + S_3. \tag{2.6c}$$

In (2.6*a*) and (2.6*b*) we successively obtain the macroscopic asymptotic states from ρ_0, ρ_1, j_0, j_1 and we rewrite the heavy (2.6*c*) relations for n_{03}, n_3:

$$\begin{aligned} -(d-1)/\sigma &= [d(2-d)n_{03}^2 + (j_0^2 - \rho_0^2)/4 + (d-1)n_{03}\rho_0] \\ &\quad /[n_{03}(\rho_0\beta + \epsilon) - \kappa_3(\alpha\rho_0^2 + \eta\rho_0) - S_3 - \kappa_3\zeta] \\ &= [n_3(n_3 + 2n_{03})d(2-d) + j_1(j_1 + 2j_0)/4 - \rho_1(\rho_1 + 2\rho_0)/4 + (d-1) \\ &\quad ((\rho_0 + \rho_1)n_3 + \rho_1 n_{03})]/[n_3((\rho_0 + \rho_1)\beta + \epsilon) \\ &\quad + n_{03}\beta\rho_1 - \kappa_3\rho_1(\alpha(\rho_1 + 2\rho_0) + \eta)]. \end{aligned} \tag{2.6c'}$$

The positivity problem for the six $n_{0i}, n_{0i} + n_i, i = 1, 2, 3$ is not trivial and an analytic study is carried out in appendix A.1. We begin with inequalities for the nonconservative parameters sufficient for two positive ρ_{as} states. For the other proofs, for brevity we restrict ourselves to the partially symmetric nonconservative $d = 2$ models with $\kappa_1 = \kappa_2, \kappa_3 = 0$. We continue with sufficient conditions so that $\rho_{as} \pm j_{as}$ is positive and we obtain: $|S_1 - S_2| \ll 1$, δ finite. Finally assuming $|S_3| \ll 1$, σ finite, we show the positivity for $n_{as,j}$, $j = 3, 1, 2$.

Our second assumption is that we are able to reduce the nonlinear differential system (1.6)–(2.5) to a linear one and satisfy the compatibility conditions. For this we assume the existence of a common $D(z = x - \xi t)$. The densities N_i are directly obtained for N_3 and by linear combination for the following equations

$$\begin{aligned}
&N_k(z) = n_{0k} + n_k/D(z) \qquad k = 1, 2, 3\\
&\rho(z) = \rho_0 + \rho_1/D(z) \qquad j(z) = j_0 + j_1/D(z)\\
&N_i(z) = [\rho(z) + (-1)^{i+1} j(z) - 2(d-1)N_3(z)]/2 = n_{0i} + n_i/D(z) \qquad i = 1, 2.
\end{aligned} \tag{2.7a}$$

We substitute into (1.6)–(2.5) the R_ρ, R_j, R_3 with (2.6a)–(2.6c) proportional to $(1-D)/D^2$ and L_ρ, L_j, L_3 to $1/D^2$ we will obtain three ODEs for D:

$$\begin{aligned}
&D^2 L_\rho = (\xi\rho_1 - j_1)\frac{\mathrm{d}D}{\mathrm{d}z} = D^2 R_\rho = (\alpha - \beta)\rho_1^2(1-D)\\
&D^2 L_j = (\xi j_1 - \rho_1 + 2(d-1)n_3)\frac{\mathrm{d}D}{\mathrm{d}z} = D^2 R_j = ((\kappa_1 - \kappa_2)\rho_1^2 - j_1\rho_1\beta)(1-D)
\end{aligned} \tag{2.7b}$$

and a similar heavy relations $L_3 D^2 = R_3 D^2$ that for brevity we do not write down. We define $\bar{n}_3 := n_3/\rho_1$, $\bar{j}_1 := j_1/\rho_1$ and write the conditions $A_\rho = A_j = A_3$ so that the three ODEs for the same function $D(z)$ are the same and we integrate easily:

$$\begin{aligned}
D(z) &= 1 + \mathrm{e}^{\gamma z}\\
A_\rho &= (\xi - \bar{j}_1)/(\alpha - \underline{)} = A_j = [\bar{j}_1\xi - 1 + 2(d-1)\bar{n}_3]/[(\kappa_1 - \kappa_2)\alpha - \beta\bar{j}_1] = A_3\\
&= \bar{n}_3\xi/[-\sigma(d(2-d)\bar{n}_3^2 + (\bar{j}_1^2 - 1)/4)/(d-1) - (\sigma + \beta)\bar{n}_3 + \kappa_3\alpha]\\
&= -\rho_1/\gamma.
\end{aligned} \tag{2.7c}$$

Forgetting γ we finally have to solve eight different relations in (2.6a, b, c)–(2.7c). We rewrite the relation between δ, ζ, S_k:

$$2(d-1)S_3 + \zeta = \delta - (S_1 + S_2). \tag{2.7d}$$

We discuss the construction, from arbitrary parameters, of the solutions and verify the above scaling invariance. From the ratios of $\alpha, \beta, \eta, \delta, \zeta, S_i, \sigma, \gamma$ by ϵ with $\kappa_i, \xi, \rho_{as}, j_{as}, n_{as,j}$ fixed, then the (2.6a)–(2.7c) relations are invariant and we can choose $\epsilon = 1$.

First for the asymmetric models (κ_i different) we choose, as arbitrary parameters, those coming from the nonconservative terms: $\alpha, \beta, \eta, \epsilon = 1, S_i, \kappa_i, \zeta$ plus the wavespeed $\xi \in (-1, 1)$. From (2.6a) and (2.6b) we deduce ρ_0, ρ_1, j_0, j_1 and from the first $A_\rho = A_j$ (2.7c) relation obtain n_3. From the second and third (2.6c') terms we obtain a cubic ($d = 3$) or quadratic ($d = 2$) n_{03} polynomial, deduce n_{03} while σ is obtained with the first term. It remains only the compatibility relation $A_\rho = A_3$ in (2.7c), without a new unknown parameter, leading to a constraint relation between our arbitrary parameters, for instance for σ.

Secondly for the partially symmetric, $\kappa_1 = \kappa_2 \neq \kappa_3$ or completely symmetric $2d\kappa_i = 1$ models we are interested in different physical problems, keeping always $\epsilon = 1$.

(i) We start with five nonconservative fixed parameters $\alpha, \beta, \eta, \delta, S_1 - S_2$ values which determine the four asymptotic macroscopic states (see (2.6a) and (2.6b)) ρ_{as}, j_{as}. We wish to know the possible associated microscopic states, the wavespeed $\xi \in (-1, 1)$ and the restrictions on the sources, sinks S_i and $\zeta > 0$ which are not fixed, satisfy a first (2.7d) relation with $2(d-1)S_3 + 2S_1 + \zeta$ known. For the four remaining relations (2.6c')–(2.7c) we add another arbitrary parameter, the elastic cross section σ. From (2.7c) we find $\xi = a_0 + a_1\bar{n}_3$ linear (a_0 and a_1 known), deduce a cubic ($d = 3$) or quadratic ($d = 2$) polynomial for $\bar{n}_3$ that, for brevity, we do not write these down. We obtain both ξ, n_3

function of the arbitrary parameters. It remains the two relations ($2.6c'$) both give n_{03} (and we can deduce all asymptotic microscopic states $n_{as,j}$, $j = 1, 2, 3$) and a second relation $S_3 + \kappa_3 \zeta$ for the two nonconservative parameters $S_3, \kappa_3 \zeta$. We can deduce both $S_3, \zeta > 0$ as functions of another arbitrary parameter S_1 which > 0 is a source or < 0 a sink. Finally, all the parameters are known and all relations satisfied with $\alpha, \beta, \eta, \delta, S_1 - S_2, \sigma, S_1$ arbitrary values.

In the simplified case $\kappa_3 = 0$, we obtain S_3 directly from ($2.6c'$) it remains the first nonconservative parameter relation ($2.7d$) with only one arbitrary parameter either S_1 or $\zeta > 0$.

(ii) We note that with five parameters $\alpha, \beta, \eta, \delta, S_1 - S_2$ we determine only four macroscopic asymptotic states. Conversely in appendix A.2 we show that from ρ_{as}, j_{as} we determine the β and $S_1 - S_2$ parameters while η and δ depend linearly on α.

Let us start with the six parameters $\rho_{as}, j_{as}, \sigma, S_3$ given and seek both all microscopic $n_{as,k}$, $k = 1, 2, 3$ states and the other nonconservative parameters. First from ($2.6a$)–($2.6b$) we deduce (appendix A.2) $\beta, S_1 - S_2$ while η, δ depend linearly on α which is unknown. For simplicity we consider the case $\kappa_3 = 0$. Secondly from ($2.6c'$) we obtain successively n_{03}, n_3 as roots of polynomials (quadratic if $d = 3$, linear if $d = 2$). Thirdly from ($2.7c$) we obtain successively ξ and α rational in ξ. From α we deduce both η, δ with the above linear relations found from ($2.6a$) and ($2.6b$). Finally it remains ($2.7d$) where $2S_1 + \zeta$ is known and we consider S_1 as arbitrary, from which we deduce the last parameters S_2, ζ. A similar heavier analysis can be performed for $\kappa_3 \neq 0$.

The important point is that when the macroscopic asymptotic states ρ_{as}, j_{as}, the cross sections σ and the source (or sink) S_3 are fixed all microscopic states are found while, due to the only knowledge of $S_1 - S_2$, different interpretations on the sources or sinks S_i, $i = 1, 2$ are possible.

2.3. Numerical calculations for a partially symmetric model $\kappa_1 = \kappa_2 = \frac{1}{2}, \kappa_3 = 0$

(i) As a first stage, in figures 1(a) and (b) for $d = 2$ and figure 1(c) for $d = 3$, for the same macroscopic asymptotic states (mass and momentum) determined from $\alpha, \beta, \eta, \epsilon = 1, \delta, S_1 - S_2$, we present the $\rho(z)$, $N_i(z)$ positive curves:

$$\begin{aligned}
&\alpha = \tfrac{5}{9} \simeq 0.55 > \beta = \tfrac{22}{45} \simeq 0.48 \qquad \eta = \tfrac{2}{90} < \epsilon \qquad \delta = \tfrac{32}{9} \simeq 3.55 \\
&S_2 - S_1 \simeq 11.55 \to \rho_0 = 8.0 \qquad \rho_1 = -\tfrac{4}{3} \qquad \rho_0 + \rho_1 > 0 \\
&j_0 = -2.35 \qquad j_1 = -0.36
\end{aligned} \tag{2.8}$$

associated to three cross section values giving for S_3 either a source or sinks:

$$\left| \begin{array}{ccccccccc}
 & \sigma & S_3 & \xi & 2S_1 + \zeta & n_{03}, n_3 & n_{02}, n_2 & n_{01}, n_1 & \gamma \\
1a & 0.73 & 5.0 & 0.57 & -18.0 & 1.46 - 0.168 & 3.7 - 0.32 & 1.36 - 0.67 & 29.6 \\
1b & 10.5 & -94.0 & 0.26 & 180.0 & 0.675 - 0.623 & 4.50.136 & 2.15 - 0.22 & -29.0 \\
1c & 4.38 & -20.2 & 0.27 & 73.0 & 0.31 - 0.308 & 4.50.13 & 2.2 - 0.23 & 126.0
\end{array} \right| . \tag{2.9}$$

In figure 1(a) all macroscopic and microscopic quantities correspond to $S_3 > 0$, source and, due to the restriction $\zeta = -18.0 - 2S_1 \geqslant 0$, to $S_1 \leqslant -9.0$, sink and $S_2 \leqslant 2.5$, source or sink. Similarly, in figure 1(b) and 1(c) we have $S_3 < 0$, sink and, due to the restriction $\zeta = 180.0 - 2S_1 \geqslant 0$ ($\zeta = 73.0 - 2S_1 \geqslant 0$) we obtain $S_1 \leqslant 90.0$ ($\leqslant 36.5$), source or sink and $S_2 \leqslant 101.5$ ($\leqslant 25.0$), source or sink. In figures 1(a)–(c) the asymptotic limits for the masses ρ_{as} and momentum j_{as} are the same but the γ are different (with a sign in figure 1(b) opposite the figures 1(a)–(c)) and the presented curves $\rho(z)$ are different.

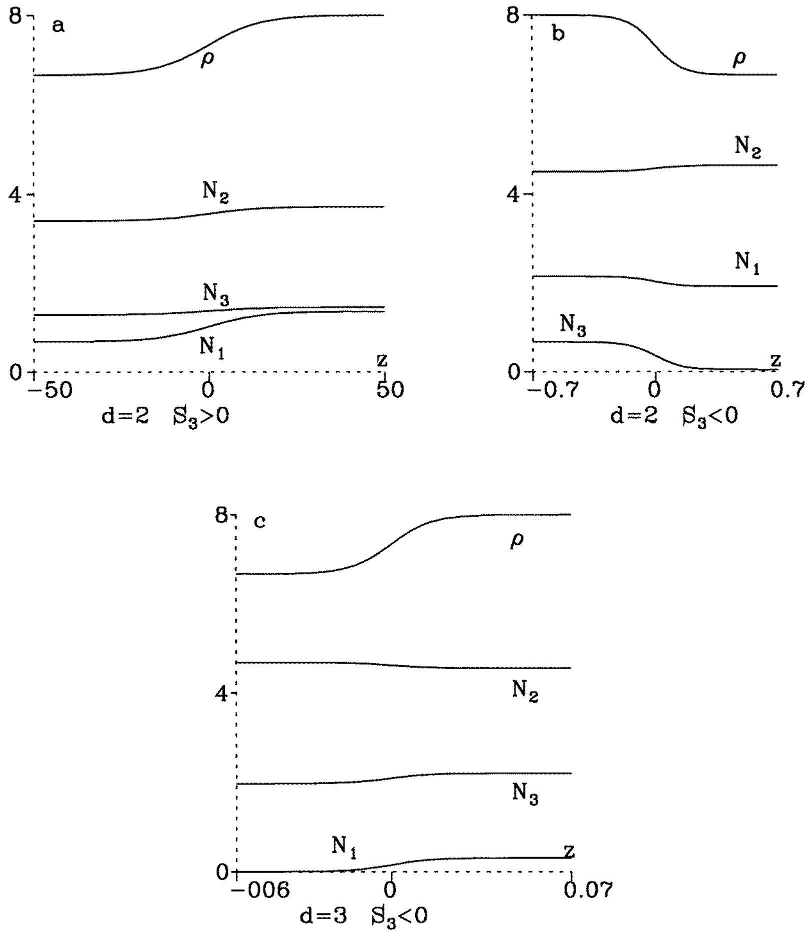

Figure 1. Travelling waves (similarity solutions) with positive asymptotic states for the nonconservative two- and three-dimensional Broadwell models with annihilation, creation of test particles. The asymptotic states for the mass and the momentum are the same. S_1, S_2 can be either source or sink but $S_1 - S_2$ is fixed. (a) $S_3 > 0$ source, (b) and (c) $S_3 < 0$ sink.

(ii) In the second stage we start for $d = 2$ with the ρ_{as}, j_{as} values written down in the second line (2.8) and the cross section $\sigma = 0.73$ as in figure 1(a), (2.9). In view of the above discussed (ii) case only S_3 is arbitrary. While β and $S_1 - S_2$ have the same (2.8) values, in contrast η and δ are linear in α and are not determined. Due to positivity, $\eta > 0$ and $|\xi| < 1$ we obtain from $S_3 \in (2.62, 5.03)$ for $\alpha, \delta, \eta, n_{as,i}, i = 1, 3$ closed intervals:

$$\alpha \in (0.32, 0.57) \qquad \delta \in (-9.0 - 3.6) \qquad \eta \in (3.47, 0.001) \qquad \xi \in (-1, 0.57)$$
$$2S_1 + \zeta \in (-26.0, -18) \qquad n_{03} \in (1.23, 1.461) \qquad n_{03} + n_3 \in (1.03, 1.29)$$
$$n_{02} \in (3.94, 3.71) \qquad n_{02} + n_2 \in (3.66, 3.4) \qquad n_{01} \in (1.58, 1.36)$$
$$n_{01} + n_1 \in (0.948, 0.684).$$

We see that S_1 arbitrary $> (13.0, 8)$ is a source. If S_3 is fixed we obtain only one solution (except for S_1, S_2) and as illustration give an example with $S_3 = 3$:

$$\alpha = 0.35 \qquad \eta = 3.03 \qquad \delta = -7.4 \qquad \xi = -0.65$$

$$n_{as,3} = 1.27, 1.07 \qquad n_{as,2} = 3.9, 3.6 \qquad n_{as,1} = 1.5, 0.9.$$

3. (1 + 1)-dimensional and periodic solutions

3.1. General (1.1) system

We rewrite (1.1), L_j associated to only N_j and in R_j we distinguish one N_k with k fixed:

$$L_j = (a_j\partial_x + b_j\partial_t)N_j \equiv R_j = N_k^2 c_k^{(j)} + N_k d_k^{(j)} + e_k^{(j)} \tag{3.1}$$

with $c_k^{(j)}$ constant and $e_k^{(j)}, d_k^{(j)}$ respectively at most quadratic, linear in the $N_i, i \neq k$. In contrast to the similarity solutions, with only one variable, for (1+1)-dimensional solutions, we have not found for the nonconservative Broadwell models (as we shall see later) the possibility to satisfy the ansatz (1.4) with more than one density. Consequently we present a method for a possible determination of (1 + 1) solutions with only one $N_k(x, t)$, the other being constants. For the applications it will remain to verify whether or not the assumptions can be justified.

Our first assumption is that only $N_k(x, t)$ while the others are constants $N_i = n_{0i}, i \neq k$ (positive for Boltzmann densities). This implies $L_j = R_j \equiv 0, j \neq k$ and the coefficients of the quadratic N_k polynomials must be zero. We also assume that $R_k = 0$ leads to two possible real roots $n_{as,k} = n_{0k}, n_{0k} + n_k$ (with at least one positive for Boltzmann densities)

$$j \neq k : c_k^{(j)} = d_k^{(j)} = e_k^{(j)} = 0 \tag{3.2a}$$

$$n_{as,k}^2 c_k^{(k)} + n_{as,k} d_k^{(k)} + e_k^{(k)} = 0 \qquad n_k c_k^{(k)} = -(2n_{0k}c_k^{(k)} + d_k^{(k)}). \tag{3.2b}$$

The constants of the quadratic (3.2*a*) polynomial must be such that the roots are real.

Our second assumption is still for $L_k = R_k$:

$$N_k(x, t) = n_{0k} + n_k N(x, t) \qquad N \equiv 1/D(x, t) \tag{3.2c}$$

which, taking into account the opposite signs of the last (3.2*b*) relation, still gives R_k proportional to $N(1 - N)$ or $(1 - D)/D^2$ and only one equation of the (1.4) type:

$$\begin{aligned}(a_k\partial_x + b_k\partial_t)N(x, t) &= c_k^{(k)} n_k N(1 - N)\\ (a_k\partial_x + b_k\partial_t)D(x, t) &= c_k^{(k)} n_k (D - 1).\end{aligned} \tag{3.2d}$$

The nonlinear PDE for N or equivalently the linear PDE for D are well known.

In the following we apply this general formalism to the Broadwell models and construct periodic and (1 + 1)-dimensional solutions. We have to verify the compatibility between the (3.2*a*) and (3.2*b*) conditions, in particular the positivity properties. In contrast to the similarity solutions, we do not need the two $n_{as,k}$ to be positive. We find a class of solutions with either only $n_{0,k}$ or $n_{0,k} + n_k$ positive, nevertheless we construct $D(x, t) > 0$ solutions such that $n_{0,k} + n_k/\sup D > 0$ if $n_{0k} + n_k > 0$ or $n_{0,k} + n_k/\inf D > 0$ if $n_{0k} > 0$.

3.2. Nonconservative Broadwell models

We choose the (1.5) system $L_j = R_j$ for N_j, $j = 1, 2, 3$ with e_j, κ_j, S_j written in (1.5):

$$\begin{aligned}&L_j = (\partial_t + e_j\partial_x)N_j = R_j = \sigma_j Q + \kappa_j(\alpha\rho^2 + \eta\rho + \zeta) - N_j(\beta\rho + \epsilon) + S_j\\ &Q = N_3^2 - N_1N_2 \qquad \rho = N_1 + N_2 + 2(d - 1)N_3, \sigma_1 = \sigma_2 = \sigma = -(d - 1)\sigma_3.\end{aligned} \tag{3.3}$$

For a solution $N_3(x, t)$, $N_i = n_{0i}, i = 1, 2$, in $R_i = 0$, the coefficients of N_3^2 must be zero, leading to $\sigma + \kappa_i\alpha = 0$ not possible from positivity $\sigma > 0, \kappa_i \geqslant 0$, $\alpha \geqslant 0$.

So our first assumption is $N_k = N_k(x,t), N_j = n_{0j}, N_3 = n_{03}$ with k, j either $1, 2$ or $2, 1$ and the possibility to find $n_{0j} > 0, n_{03} > 0$ and at least one $n_{as,k} > 0$ for N_k.

We have $L_3 = L_j \equiv 0$ and from $R_j(N_k) = R_3(N_k) \equiv 0$ we obtain two quadratic $N_k(x,t)$ polynomials with coefficients which must be zero. The coefficient of N_k^2 in $R_j = 0$ being $\kappa_j\alpha$ we obtain either $\kappa_j = 0$ or α. In the first case the coefficient of N_k in $R_j = 0$ leads to $\beta + \sigma = 0$ which violates positivity, so that the restriction $\alpha = 0$ remains.

From the vanishing coefficients of N_k in $R_3 = R_j \equiv 0$ (see the linear N_k polynomials in appendix A.3), we obtain two other relations leading to positive densities n_{03}, n_{0j}:

$$\begin{aligned} &\alpha = 0 \qquad n_{0j} = \kappa_j\eta/(\sigma+\beta) > 0 \\ &n_{03} = (\eta/\beta)[\kappa_3 + \sigma\kappa_j/(d-1)(\sigma+\beta)] > 0. \end{aligned} \tag{3.4a}$$

It remains the constant terms which, with (3.4a), can be written:

$$\sigma[n_{03}^2 + n_{0j}^2 + 2(d-1)n_{03}n_{0j}] = n_{0j}\epsilon - S_j - \kappa_j\zeta = (d-1)(S_3 + \kappa_3\zeta - n_{03}\epsilon). \tag{3.4b}$$

For the $R_k(n_{as,k}) = 0$ relation we write the quadratic $n_{as,k}$ polynomial:

$$\begin{aligned} &n_{as,k} = n_{0k}, n_{0k} + n_k \qquad a_1 := (\kappa_j - \kappa_k)\eta + 2(d-1)n_{03}\beta + \epsilon \\ &a_0 := \kappa_k(\zeta + \eta(n_{0j} + 2(d-1)n_{03}) + \sigma n_{03}^2 + S_k \\ &\beta n_{as,k}^2 + n_{as,k}a_1 - a_0 = 0 \qquad \beta(n_k + 2n_{0k}) + a_1 = 0. \end{aligned} \tag{3.4c}$$

We can verify the above scaling invariance with the ratios of $\beta, \eta, \zeta, \sigma, S_i$ by $\epsilon > 0$ with κ_i fixed, then in (3.4a)–(3.4c), $n_{0j}, n_{03}, n_{as,k}$ are invariant and we can choose $\epsilon = 1$.

From (3.4c) we deduce n_{0k}, n_k and two different classes.

(i) If either $S_k > 0$ (source) with $a_0 > 0$ or if $\kappa_j > \kappa_k$ (or symmetric model $\kappa_j = \kappa_k$) with $a_1 > 0$, necessarily one of the two possible asymptotic states $n_{0k}, n_k + n_{0k}$ is negative and we must construct solutions with only one asymptotic state.

(ii) In contrast if S_k is sufficiently negative (sink) with $a_0 < 0$ and $\kappa_j < \kappa_k$ (asymmetric model), we can have (see appendix A.4) two positive asymptotic states.

For our second assumption we write $N_k = n_{0k} + n_k N(x,t)$, substitute into $L_k = R_k$, take into account the second (3.4c) relation, deduce a nonlinear PDE for $N(x,t)$

$$(\partial_t + e_k\partial_x)N(x,t) = -\beta n_k N^2 - N(\beta 2n_{0k} + a_1) = -\beta n_k N(N-1)$$

or equivalently a PDE for $D = N^{-1}$. We write also the mass ρ:

$$\begin{aligned} &N_k(x,t) = n_{0k} + n_k/D(x,t) \qquad \rho = n_{0k} + n_{0j} + 2(d-1)n_{03} + n_k/D \\ &(\partial_t + (-1)^{(k+1)}\partial_x)D(x,t) = \beta n_k(1-D). \end{aligned} \tag{3.5}$$

Before discussing the solutions of the PDE (3.5), we consider, from relations (3.4a)–(3.4c), the determination of the parameters from arbitrary ones.

We start with $\sigma, \beta, \eta, \epsilon, \kappa_1, \kappa_2, S_k, \zeta$, deduce $\kappa_3 = -(\kappa_1+\kappa_2)/2(d-1)$ and successively $n_{0j} > 0, n_{03} > 0$, (3.4a) S_j, S_3, (3.4b) and n_k, n_{0k} (3.4c) with at least one n_{0k} or $n_{0k} + n_k$ positive. From the superposition principle we can add any number of solutions satisfying the linear PDE (3.5) for $D(x,t)$.

We begin with the periodic solutions and, for brevity, write one of the simplest

$$\begin{aligned} &N_k = n_{0k} + n_k/D \qquad D(x,t) = 1 + \Delta(x,t) \\ &\Delta(x,t) = \overline{d}\mathrm{e}^{-n_k\beta t}\cos\gamma(x + (-1)^k t) \qquad \overline{d} > 0, \gamma \text{ arbitrary}. \end{aligned} \tag{3.6}$$

We could also have for the oscillating terms $\sum d_m \cos(\gamma_m(..)) + \overline{d}_m \sin(\overline{\gamma}_m(.))$ with m, d_m, $\overline{d}_m, \gamma_m, \overline{\gamma}_m$ arbitrary, etc. The oscillations propagate with time and the interesting solutions are those with $n_k > 0$ because the damping factor when $t \to \infty$ leads to the asymptotic state $n_{0k} + n_k$ for which we can, in (3.4c), choose a positive $n_{as,k} > 0$ root.

We prove that we can always have a damping factor $n_k > 0$ and positive solutions.

(i) If only one of the two roots $n_{0k}, n_{0k} + n_k$ is positive (as shown above S_k source or $\kappa_j > \kappa_k$) we choose $n_{0k} < 0$, $n_{0k} + n_k > 0$ and necessarily $n_k > 0$. Furthermore, $N_k > n_{0k} + n_k/(1+\bar{d})$ and we can choose $\bar{d}$ sufficiently small such that $n_{0k} + n_k/(1+\bar{d}) > 0$ or $N_k, \rho > 0$ for $t \geqslant 0$ and $|x| \in (0, +\infty)$.

(ii) If the two roots $n_{as,k}$ are positive (S_1 sink, $\kappa_k > \kappa_j$ and appendix A.4 for sufficient conditions), with $\bar{d} < 1$ we obtain $D > 0$ and the positivity is satisfied. In appendix A.4 we show that we can always choose $n_k > 0$. In both cases with $n_k > 0$, positivity at $t = 0$ gives positivity $\forall t > 0$ and we see only one asymptotic state $n_{0k} + n_k$ when $t \to \infty$.

We continue with the $(1 + 1)$-dimensional solutions. Here also we can start with an arbitrary number $\Delta = \sum d_m \mathrm{e}^{\gamma_m (x - \xi_m t)}$ of terms but, for simplicity, we consider only two of them that we substitute into the linear PDE (3.5)

$$N_k = n_{0k} + n_k/D \qquad D = 1 + \Delta \qquad \Delta(x,t) = \sum_{1}^{2} d_m \mathrm{e}^{\gamma_m (x - \xi_m t)} \geqslant 0$$
$$\gamma_m = \beta n_k/(\xi_m + (-1)^k) \qquad \text{with arbitrary } d_m > 0, \xi_m. \tag{3.7}$$

We prove positivity in the two cases with only one $n_{as,k}$ positive or two.

(i) Choosing γ_m such that $\gamma_1\gamma_2 < 0$, for t fixed $D \to \infty$, $N_k \to n_{0k}$ when $|x| \to \infty$ and we discuss the nontrivial case when $n_{0k} > 0$ but $n_{0k} + n_k < 0$. Furthermore, choosing $\gamma_m \xi_m < 0$ for at least one of the two m values, it follows for $|x|$ fixed and $t \to \infty$ that still $D \to \infty$, $N_k \to n_{0k} > 0$. The important point is that Δ cannot vanish and we must choose the arbitrary positive parameters d_m such that $n_{0k} + n_k/(1+\inf\Delta) > 0$ leading to $N_k > 0$.

(ii) If $\gamma_1\gamma_2 > 0$ then, like for similarity solutions, $D \to \infty, 1$ when $|\gamma_m x| \to \infty$ and the two states $n_{0k}, n_{0k} + n_k$ must be positive. Due to the fact that $\Delta \geqslant 0$, $D \geqslant 1$, the positivity is always satisfied but the patterns will be different because N_k will have the two state limits. With the two $n_{as,k} > 0$ and $\gamma_1\gamma_2 < 0$ we could also construct positive solutions.

In appendix A.5 we explain the difficulty to obtain $(1 + 1)$ solutions of the (1.4) type $N_i(x,t) = n_{0i} + n_i N(x,t)$ with more than one density. The possible solutions are self-similar (as in section 2) or one-dimensional but not $(1 + 1)$.

3.3. Numerical calculations

For $(1 + 1)$ solutions we have two cases, for the positivity associated to only one positive $n_{as,k}$ or two when $|x| \to \infty$ and t finite. The solutions are very different (figures 2(b) and 3(b)). However, for oscillating damped solutions, only one asymptotic state really exists, the one with $t \to \infty$ and the solutions are similar (figures 2(a) and 3(a)).

We first discuss the case, for a symmetric model $\kappa_i = 1/2d$, with only one positive $n_{as,k}$. In figures 2(a) and (b) with $d = 3$ we choose $k = 1$ or $N_1(x,t)$, $N_2 = n_{02}$, $N_3 = n_{03}$ and:

$$S_1 = 2.28 \text{ source} \qquad \beta = 0.628 \qquad \sigma = 0.028 = \eta \qquad \epsilon = 1 \qquad \zeta = 0.077.$$

In figure 2(b) we choose ξ_m such that $\gamma_m \xi_m < 0$, $d_m = 1.6$ and deduce from (3.4a)–(3.4c)

$$n_{02} = 0.05 \qquad n_{03} = 0.09 \qquad S_2 = 0.028 \qquad S_3 = 0.077 \qquad n_{as,1} = -0.312, 1.18$$

figure 2(a) : $n_{01} = -0.312 \qquad n_1 = 4.3 \qquad n_{01} + n_1 = 1.18 > 0 \qquad n_1\beta = 2.76$
$\qquad \bar{d} = 0.15$

figure 2(b) : $n_{01} + n_1 = -0.312 \qquad n_1 = -4.3 \qquad n_{01} = 1.18 > 0 \qquad \gamma_1 = 6 > 0$
$\qquad \gamma_2 = -24.6.$

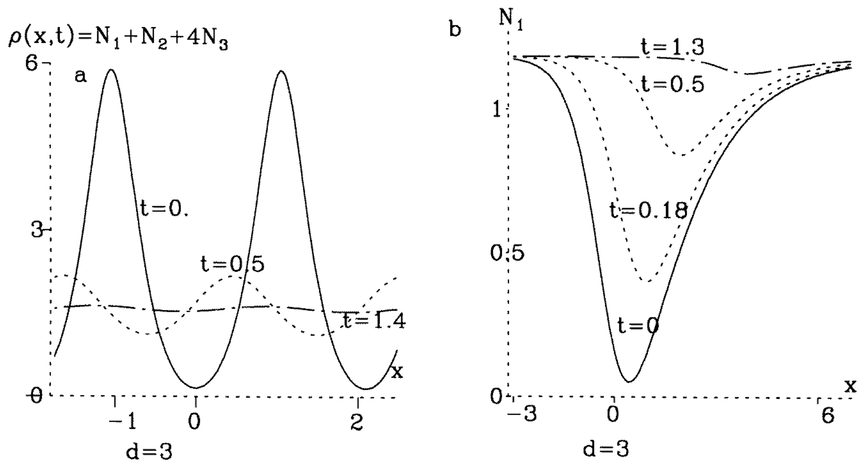

Figure 2. Periodic and $(1+1)$-dimensional solutions for the nonconservative $d = 3$ Boltzmann symmetric model with annihilation, partly creation of test particles and sources. We notice a source S_1 and only one positive asymptotic state. (a) Damped periodic waves; (b) $(1 + 1)$ solution relaxing towards a constant.

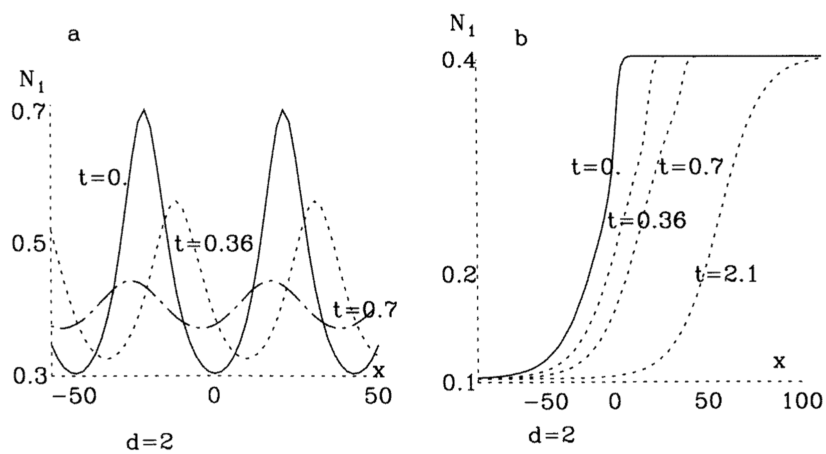

Figure 3. Periodic and $(1 + 1)$-dimensional solutions for an asymmetric $d = 2$ model and S_1 a sink. (a) Damped periodic waves with one positive asymptotic state; (b) $(1 + 1)$ solution with two positive asymptotic states.

The positive solutions are respectively periodic damped in figure 2(a) and $(1 + 1)$-dimensional in figure 2(b). In figure 2(b) we also have in fact some damping because, when $t \to \infty$, we see a limit for N_1 which is the same as the $\sup N_1 = n_{01}$ limit when $|x| \to \infty$ with t finite. In contrast $\inf N_1$ depends on both x, t finite but still tends to n_{01} when $t \to \infty$.

Secondly we present solutions for an asymmetric model $\kappa_1 = 0.9 \gg \kappa_2 = \kappa_3 = \frac{1}{30}$ with two positive $n_{as,k}$ in figures 3(a) and (b), $d = 2$, with $N_1(x, t)$, $N_2 = n_{02}$, $N_3 = n_{03}$ and

$$S_1 = -\tfrac{19}{18} \simeq -1.05 \text{ sink} \qquad \epsilon = 1 \qquad \beta = \tfrac{2}{3}$$
$$\sigma = 0.005\,55 \qquad \eta = 0.555 \qquad \zeta = \tfrac{7}{18} = 0.0388.$$

In figure 3(b) we choose ξ_m such that $\gamma_m\xi_m > 0$, $d_m = 1.6$ and deduce from (3.4a)–(3.4c):

$n_{02} = 0.025 \qquad n_{03} = 0.03 \qquad S_2 = \frac{1}{900} \simeq 0.0011 \qquad S_3 \simeq 0.0019$

figure 3(a) : $n_{01} = 0.1 \qquad n_1 = 0.3 > 0 \qquad n_{01} + n_1 = 0.4 > 0 \qquad n_1\beta = 0.2$

$\bar{d} = 0.25$

figure 3(b) : $n_{01} + n_1 = 0.1 \qquad n_1 = -0.3 < 0 \qquad n_{01} = 0.4 > 0 \qquad \gamma_1 = \frac{4}{9}$

$\gamma_2 = \frac{11}{35} \simeq 0.31.$

In figure 3(a) with periodic damped solution and asymptotic state $n_{01} + n_1 > 0$ when $t \to \infty$, the features are similar to those in figure 2(a). In figure 3(b) (in contrast to figure 2(b)) we see the two different asymptotic limits $n_{01}, n_{01} + n_1$ when $|x| \to \infty$ and the limit $n_{01} + n_1$ when $t \to \infty$ and $|x|$ finite. Roughly speaking the t-dependent solutions are translated, when the time increases, although they are not self-similar.

Due to the scaling invariance, if we replace $\epsilon = 1$ by a value larger or smaller than 1, then βn_1 becomes larger or smaller and the damping in figures 2(a)–3(a) becomes stronger or weaker.

4. Conclusion

For the similarity solutions with two asymptotic states we were interested in the classical inverse problem of the possible determination of the microscopic states (here $n_{as,j}$, $j = 1, 2, 3$) from the knowledge of the macroscopic ones (here ρ_{as} for the mass and j_{as} for the momentum). If we only give ρ_{as}, j_{as} we obtain a two-parameter family of $n_{as,j}$. If we add the elastic cross section σ we are reduced to a one-parameter family. Finally, with ρ_{as}, j_{as}, σ and the knowledge of a source (or sink) S_3 associated to the density with momentum zero, then we obtain only one class of $n_{as,j}$. However, for the other two sources (sinks), for the velocities ± 1, only $S_1 - S_2$ is known so that they can be either sources or sinks.

For the $(1 + 1)$-dimensional solutions with either only one asymptotic state or two we were interested to understand this distinction. We find either a source or a strong sink for one S_i, $i = 1, 2$ and a dominant associated κ_i.

Recently, applying the method presented in sections 2 and 3 for $p = 2$, we have determined the same classes of similarity, periodic, $(1 + 1)$ solutions for the different Carleman, McKean, Illner [7] two velocity models. In principle, the same method for p independent densities could lead, due to the above discussion on the number of constraints and parameters, to solutions similar to the present $p = 3$ ones, but it will remain to verify that both the positivity properties and the compatibility between the constraints can be satisfied.

The presented similarity solutions are not too different for conservative and nonconservative models. For a model with p independent densities and q conservation laws, we obtain for conservative model a compatibility between only $p-q$ scalar Riccati equations and between p for the nonconservative one. However, the other class of nonmonotonic [6] similarity solutions, obtained for the conservative models, have, up until now, not been found for the nonconservative ones and it will be interesting to seek such solutions. We notice that the formalism is more complicated with more constraint relations. In view of the standard Riccati coupled solutions [8], mainly projective and conformal, they have, up until now, not been found in both conservative and nonconservative models.

In the nonconservative two-velocity models [2], the determination of the hydrodynamic-associated equations was performed and it seems useful to generalized to other nonconservative models. For nonconservative models it will be useful to study exact solutions satisfying boundary conditions [2] which is outside the scope of this paper.

The periodic and $(1+1)$-dimensional solutions found are very different for conservative and nonconservative models. It seems useful to try to apply the method presented to other nonconservative models with $p > 3$ or more than three independent densities.

As usual the exact solutions can be a paradigm for a study of numerical solutions around them. For instance for the $(1+1)$-solutions with only one x, t-dependent density, it will be useful to determine numerically more general solutions in order to understand what happens when they reduce to the exact ones.

Acknowledgment

I thank Professor T Platkowski who asked me to find exact solutions for the nonconservative models.

Appendix. Nonconservative Broadwell models

A.1. Positivity for the similarity solutions

We begin with (2.6a): $(\alpha - \beta)\rho_{as}^2 + \delta = (\epsilon - \eta)\rho_{as}$, recall at least $\beta > 0, \eta > 0, \zeta > 0$, choose $\epsilon = 1$ and write sufficient (not necessary) conditions for two real positive ρ_{as} roots:

$$\alpha > \beta \qquad 1 > \eta \qquad \delta = S_1 + S_2 + 2(d-1)S_3 + \zeta > 0 \qquad (1-\eta)^2 > 4(\alpha - \beta)\delta. \tag{A.1}$$

We choose the model $\kappa_1 = \kappa_2$ in (2.6b) with $\Lambda := \beta\rho_{as} + 1$, $j_{as} = (S_1 - S_2)/\Lambda$, assume $|S_1 - S_2| \ll 1$ and δ finite and obtain from (2.6a) and (2.6b):

$$\Omega^{\pm} := \rho_{as} \pm j_{as} = [\alpha\rho_{as}^2 + \rho_{as}\eta + \delta \pm (S_1 - S_2)]/\Lambda > \delta - |S_1 - S_2| > 0. \tag{A.2}$$

In (2.6c) and (2.6c') we choose the model $d = 2$, $\kappa_3 = 0$, assume $|S_3| \ll 1$, σ finite and obtain:

$$\tilde{n}_{as,3} = n_{as,3}(\Lambda + \rho_{as}\sigma) = S_3 + \sigma\Omega^+\Omega^-/4 > S_3 + \sigma(\delta - |S_1 - S_2|)^2/4 > 0. \tag{A.3}$$

For the $n_{as,i}$, $i = 1, 2$ we define $n_{as}^{\pm} = \Omega^{\pm}/2 - n_{as,3}$ and obtain:

$$\begin{aligned} 2\tilde{n}_{as}^{\pm} &= 2n_{as}^{\pm}(\rho_{as}\sigma + \Lambda) = -2S_3 + \Omega^{\pm}(\rho_{as}\beta + 1 + \sigma\Omega^{\pm}/2) \\ 2\tilde{n}_{as}^{\pm} &> -2S_3 + (\delta - |S_1 - S_2|)(1 + \rho_{as}\beta) + \sigma(\delta - |S_1 - S_2|)^2/2 > 0. \end{aligned} \tag{A.4}$$

In conclusion for the $d = 2$, $\kappa_1 = \kappa_2, \kappa_3 = 0$ models, with the assumptions $|S_1 - S_2| \ll 1$, $|S_3| \ll 1$ and δ, σ finite, the six asymptotic states $n_{as,j}$, $j = 1, 2, 3$ are positive.

A.2. Relations between ρ_{as}, j_{as} and the nonconservative parameters

From ρ_{as}, j_{as} in (2.6a) and (2.6b) we obtain β, $S_1 - S_2$ and a linear relation between α, η, δ:

$$\begin{aligned} &\epsilon = 1, \kappa_1 = \kappa_2 : A := j_1(\rho_0 + \rho_1) + j_0\rho_1 \\ &\beta = -j_1/A \qquad S_1 - S_2 = j_0\rho_1(j_1 + j_0)/A \\ &\alpha = -j_1/A + \delta/\rho_0(\rho_0 + \rho_1) = [-\eta + (\rho_1 j_0 - j_1\rho_0)/A]/(\rho_1 + 2\rho_0). \end{aligned} \tag{A.5}$$

A.3. Linear R_j, R_3 polynomials for the (1 + 1) solutions

We write the (3.3) $R_j(N_k) = R_3(N_k) \equiv 0$ polynomials, linear when $\alpha = 0$:

$$N_k[-(\sigma+\beta)n_{0j} + \kappa_j\eta] + \sigma n_{03}^2 + (\kappa_j\eta - \beta n_{0j})(n_{0j} + 2(d-1)n_{03})$$
$$+\kappa_j\zeta - n_{0j}\epsilon + S_j$$
$$N_k[\sigma n_{0j}/(d-1) + \kappa_3\eta - n_{03}\beta] - \sigma n_{03}^2/(d-1) + (\kappa_3\eta - \beta n_{03})(n_{0j} + 2(d-1)n_{03})$$
$$+\kappa_3\zeta - n_{03}\epsilon + S_3.$$

The linear and constant terms are zero.

A.4. Sufficient coonditions for positive $n_{as,k}$ (1 + 1) solutions

We give sufficient conditions so that the two roots of $\beta n_{as,k}^2 + n_{as,k}a_1 - a_0 = 0$, written in (3.4*c*) are positive. We choose $\epsilon = 1$, assume β large, η and $-S_k$ of the same order while the other parameters are small. More precisely:

$$\begin{aligned} &\beta \gg 1 \qquad \eta = \overline{\eta}\beta \qquad S_k = -\overline{s}_k\beta < 0 \qquad \text{with } \overline{\eta} > 0, \overline{s}_k > 0 \text{ finite} \\ &\sigma/\beta \ll 1 \qquad \kappa_i/\beta \ll 1 \qquad \zeta/\beta \ll 1 \end{aligned} \tag{A.6}$$

we deduce for n_{0j}, n_{03} and a_0, a_1 written respectively in (3.4*a*) and (3.4*b*):

$$\begin{aligned} &n_{0j} \simeq \kappa_j\overline{\eta} \qquad n_{03} \simeq \kappa_3\overline{\eta} \qquad a_1 \simeq \beta\overline{\eta}(1-2\kappa_k) \\ &a_0 \simeq \beta[\overline{\eta}^2\kappa_k(1-\kappa_k) - \overline{s}_k] \qquad \Delta/\beta^2 = \overline{\eta}^2[1 + 8\kappa_k(\kappa_k - 1) + \overline{s}_k]. \end{aligned} \tag{A.7}$$

Δ being the discriminant of the quadratic polynomial. Finally, sufficient conditions for two positive roots are provided by (A.6) and $\kappa_k > \frac{1}{2}, \overline{s}_k > 3, \overline{s}_k > \overline{\eta}^2/4$. We deduce $a_1 < 0$, $\overline{s}_k > \overline{\eta}^2\kappa_k(1-\kappa_k)$ or $a_0 < 0$ and $\overline{s}_k + 1 + 8\kappa_k(\kappa_k - 1) > 0$ or $\Delta > 0$.

For periodic solutions and the damping factor βn_k, let us define $n_{as,k}^{\pm} = (-a_1 \pm \sqrt{\Delta})/2\beta$. Choosing $n_{0k} = n_{as,k}^-$, $n_{0k} + n_k = n_{as,k}^+$ we obtain $\beta n_k = \sqrt{\Delta} > 0$.

A.5. (1 + 1) Solutions of the type (1.4) $N_j(x,t) = n_{0j} + n_jN(x,t)$

We assume that more than one $N_j(x,t)$ are of this type and the set $(n_{0j}, n_{0j} + n_j)$ are the set of two roots coming from the set $(R_j = 0)$. We rewrite (3.3):

$$\begin{aligned} &(\partial_t + e_j\partial_x)N(x,t) = N(N-1)C_j \qquad j = 1,2,3 \\ &C_j := [\sigma_j(n_3^2 - n_1n_2) + \alpha\kappa_j(n_1 + n_2 + 2(d-1)n_3)^2]/n_j - \beta(n_1 + n_2 + 2(d-1)n_3). \end{aligned} \tag{A.8}$$

If one $N_k(x,t) = n_{0k}$ is a constant, then necessarily $C_k = 0$ and we find, by linear combination of the two other, one-dimensional or similarity solutions not $(1+1)$:

$$(i) N_j(x,t),\ j = 1,2,\ N_3 = n_{03} \to 2\partial_t N = (C_1 + C_2)N(N-1),\ 2\partial_x N$$
$$= (C_1 - C_2)N(N-1) \to [(C_1 - C_2)\partial_t - (C_1 + C_2)\partial_x]N$$
$$= 0 \to N(z),\ z = (C_1 + C_2)t + (C_1 - C_2)x$$
$$(ii) N_j(x,t),\ j = 1,3,\ N_2 = n_{02} \to \partial_t N = C_3N(N-1),\ \partial_x N$$
$$= (C_1 - C_3)N(N-1) \to [(C_1 - C_3)\partial_t - C_3\partial_x]N$$
$$= 0 \to N(z),\ z = C_3t + (C_1 - C_3)x$$

and similarly for $N_j(x,t)$, $j = 2,3$, $N_1 = n_{01}$. For the three $N_j(x,t)$, coming back to two of them we obtain the previous results.

References

[1] Boffi V C, Franceschini V G and Spiga G 1985 *Phys. Fluids* **28** 3232
Boffi V C and Spiga G 1987 *TTSP* **16** 175
For a survey, see Spiga G 1989 *Lecture Notes in Mathematics* vol 1460, ed G Toscani *et al* (Berlin: Springer) p 203 and references therein
Spiga G 1990 *Mod. Math. Meth. Trans. Theory* vol 51, ed W Greenberg and J Polewczak p 294
Spiga G, Dudek G and Boffi V C 1989 *Discrete Kinetic Theory, Lattice Gases and Hydrodynamics* ed R Monaco (Singapore: World Scientific) p 315
Piechor K 1989 *Arch. Mech.* **41** 95
Caraffini G L and Spiga G 1994 *TTSP* **23** 9–25

[2] Platkowski T and Spiga G 1992 *Eur. J. Mech.* B **11** 349–62
Platkowski T and Spiga G 1994 *Physica* **210A** 155
Piechor K and T Platkowski 1996 RW 96-03

[3] Gatignol R 1975 *Lecture Notes in Physics* vol 36 (Berlin: Springer)
Cabannes H 1980 *The Discrete Bolt Equations* (Berkeley, CA:)
Platkowski T and Illner R 1988 *SIAM Rev.* **30** 213
Bellomo N and Gustafsson T 1991 *Rev. Mod. Phys.* **137**
Cercignani C 1994 *TTSP* **23** 1

[4] Cornille H 1987 *J. Phys. A: Math. Gen.* **20** 1973
Cornille H 1987 *Phys. Lett.* **125A** 253 for similarity and $(1+1)$ solutions of the Broadwell models
For general surveys see Cornille H 1989 *Partially Integrable Evolution Equations in Physics* ed R Conte (Dordrecht: Kluwer) p 39
Cornille H and Platkowski T 1992 *Progress in Astronautics and Aeronautics* vol 159, ed B Shizgal and D Weaver (AIAA) p 15
For recent papers see Bobylev A V and Spiga G 1994 *J. Phys. A: Math. Gen.* **27** 133
Cornille H and d'Almeida A 1996 *J. Math. Phys.* **37** 5476
Cabannes H 1997 *Eur. J. Mech.* B **16** and references therein

[5] Cornille H 1989 *Lecture Notes in Mathematics* vol 1460, ed G Toscani *et al* (Berlin: Springer) p 70
Ciulli S, Scheck F and Thirring W (ed) 1990 *Rigor. Meth. Part. Phys.* (Berlin: Springer) p 119
Ciulli S, Scheck F and Thirring W (ed) 1990 *Rigor. Meth. Part. Phys.* (Berlin: Springer) p 168

[6] Cornille H and Platkowski T 1992 *J. Math. Phys.* **33** 2587
Cornille H and Platkowski T 1993 *J. Stat. Phys.* **71** 719
Cornille H and Platkowski T 1994 *TTSP* **23** 75

[7] Illner R 1979 *Math. Methods Appl. Sci.* **1** 187
McKean M P 1967 *J. Comb. Theory* **2** 358

[8] Reid W T 1972 *Riccati Differential Equations* (New York: Academic)
Bountis T G, Papageorgiou V and Winternitz P 1986 *J. Math. Phys.* **27** 1215